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Exponential Convexity Method

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In this paper generalized Stolarsky and related means are recognized as an application of particulary defined linear functional on family of $n-$convex function. However, this approach leads to a more general method of constructing exponentially convex functions and means of Cauchy type.

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1. Introduction

Exponentially convex functions have been introduced by S. N. Bernstein eighty years ago in [2]. Independently, a few years later, D. V. Widder introduced these functions in [18] as a sub-class of convex functions on a given interval (a, b). It is convenient here to note that Widder in his book [19] made an excellent account of theory that in conclusion leads to integral representation of exponentially convex functions. After initial development there is a big gap in time before applications and examples of interest have been constructed. One of reasons is that, aside from absolute monotone functions and completely monotone functions as special classes of exponential convexity, to this date there is no operative criteria to recognize exponential convexity.

Stolarsky means are two parameter Cauchy means that have very well-known monotonicity property. Recently, proof of this fact was given in [6] using a particular family of convex functions in combination with log-convexity. Exponentially convex functions are log-convex and it turns out that more generalized versions of Stolarsky means contain exponentially convex functions as a basic ingredient. More precisely, we will show that generalized Stolarsky and related means can be interpreted through action of linear functional on a specific family of $n-$convex

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functions that in the end give us examples of exponential convexity. Our method leads us to the conclusion that any inequality that is expressible in a form of a linear functional and which is valid for $n-$convex functions can be used for production of exponential convexity.

Our method for construction of exponentially convex functions in this article becomes more significant especially when we read the following lines from [4]:

> There is a vast body of literature on the construction of positive definite functions, functions, both in terms of characteristic functions and in terms of covariance functions. In contrast, the literature on exponentially convex functions is minimal.

2. Exponentially convex functions

We will give definition of exponentially convex function as it was originally given by Bernstein in [2] (see also [1], [10], [11]). Here I stands for open interval in $\mathbb{R}$.

Definition 2.1. A function $\psi : I \to \mathbb{R}$ is exponentially convex on I if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(x_i + x_j\right) \geq 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i, x_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $x_i + x_j \in I$, $1 \leq i, j \leq n$.

Proposition 2.2. *Let $\psi : I \to \mathbb{R}$. The following statements are equivalent.*

(*i*) *ψ is exponentially convex on I*

(*ii*) *ψ is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1}$$

for every $\xi_i \in \mathbb{R}$ and every $x_i \in I$, $1 \leq i \leq n$.

For exponentially convex function on $\mathbb{R}$ we will just say exponentially convex function.

Remark 2.3. From (1) we have the following conclusions:

(i) if ψ is exponentially convex on I then $\psi(x) \geq 0$, for all $x \in I$; for any $c \geq 0$, $c\psi$ is again exponentially convex;

(ii) if ψ_1 and ψ_2 are exponentially convex on I, then $\psi_1 + \psi_2$ is also exponentially convex on I;

(iii) if ψ is an exponentially convex function then for any $d, t \in \mathbb{R}$, $x \mapsto \psi(dx)$ and $x \mapsto \psi(x - t)$ are exponentially convex functions.

Using basic calculus we have further conclusions.

Corollary 2.4. *If ψ is exponentially convex on I, then the matrix*

$$\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}$$

is positive semi-definite. Particularly,

$$\det\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0,$$

for every $n \in \mathbb{N}, x_i \in I,\ i = 1, \dots, n$.

Corollary 2.5. *If $\psi : I \to (0, \infty)$ is an exponentially convex function, then ψ is a log-convex function i.e. $\log\psi$ is a convex function.*

Proof. From (1), for $n = 2$, we conclude

$$\xi_1^2\psi(x) + 2\xi_1\xi_2\psi\left(\frac{x+y}{2}\right) + \xi_2^2\psi(y) \geq 0,$$

for all $\xi_1, \xi_2 \in \mathbb{R}$ and all $x, y \in I$. Hence

$$\psi^2\left(\frac{x+y}{2}\right) \leq \psi(x)\psi(y), \quad \text{for all } x, y \in I. \tag{2}$$

Since ψ is a continuous function we have

$$\psi\left(\lambda x + (1-\lambda)y\right) \leq \psi(x)^{\lambda}\psi(y)^{1-\lambda}, \tag{3}$$

for all $x, y \in I$ and any $\lambda \in [0, 1]$. □

Proposition 2.6. *Assume that ψ is a nonnegative continuous function defined on I such that*

$$\psi^2\left(\frac{x+y}{2}\right) \leq \psi(x)\psi(y), \quad x, y \in I \tag{4}$$

If $\psi(x_0) = 0$ for some $x_0 \in I$, then $\psi \equiv 0$ on I.

Proof. Choose any $y \in I$. From (4) we have

$$\psi\left(\frac{1}{2^n}x_0 + \left(1 - \frac{1}{2^n}\right)y\right) \leq \psi(x_0)^{1/2^n}\psi(y)^{1-1/2^n}. \tag{5}$$

Hence $\psi\left(\frac{1}{2^n}x_0 + \left(1 - \frac{1}{2^n}\right)y\right) = 0$ and $\psi(y) = \lim_n \psi\left(\frac{1}{2^n}x_0 + \left(1 - \frac{1}{2^n}\right)y\right) = 0$. □

Remark 2.7. From the previous proposition, we have that if an exponentially convex function at some point of its domain is equal to zero, then it is zero on the whole domain.

Remark 2.8. A function $\psi : I \to (0, \infty)$ that satisfies (4) is said to be log-convex in J-sense.

The next theorem will play an important role in the sequel.

Theorem 2.9. *Let $f : I \to (0,\infty)$ be a log-convex, derivable function. Let $M : I \times I \to (0,\infty)$ be defined by*

$$M(x,y) = \begin{cases} \left(\frac{f(x)}{f(y)}\right)^{\frac{1}{x-y}}, & x \neq y; \\ \exp\left(\frac{f'(x)}{f(x)}\right), & x = y. \end{cases} \tag{6}$$

If $x_1 \leq x_2,\ y_1 \leq y_2$ then

$$M(x_1,y_1) \leq M(x_2,y_2). \tag{7}$$

Proof. Since the function $\log f$ is convex, we have (see [13], p. 2)

$$\frac{\log f(x_1) - \log f(y_1)}{x_1 - y_1} \leq \frac{\log f(x_2) - \log f(y_2)}{x_2 - y_2}, \tag{8}$$

for $x_1 \leq x_2,\ y_1 \leq y_2;\ x_1 \neq y_1,\ x_2 \neq y_2$, concluding $M(x_1,y_1) \leq M(x_2,y_2)$. If $x_1 = y_1 \leq x_2$ we apply the limit $\lim_{y_1 \to x_1 -}$ to (8) to conclude

$$M(x_1,x_1) \leq M(x_2,y_2).$$

Other possible cases are treated similarly. □

One of the most important properties of exponentially convex functions is their integral representation.

Theorem 2.10. *The function $\psi : I \to \mathbb{R}$ is exponentially convex on I if and only if*

$$\psi(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in I \tag{9}$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof. See [1], p. 211. □

A first application of Theorem 2.10 is that exponential convexity is closed under multiplication.

Corollary 2.11. *If $\psi_1,\ \psi_2$ are exponentially convex functions on I, then $\psi_1\psi_2$ is also exponentially convex on I.*

Proof. For any $n \in \mathbb{N}$ and all $\xi_i \in \mathbb{R},\ x_i \in I,\ i = 1,\ldots,n$ we have

$$\sum_{i,j=1}^{n} \xi_i\xi_j\psi_1\left(\frac{x_i+x_j}{2}\right)\psi_2\left(\frac{x_i+x_j}{2}\right)$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{n} \xi_i e^{tx_i/2}\xi_j e^{tx_j/2}\psi_1\left(\frac{x_i+x_j}{2}\right)\sigma_2(dt) \geq 0,$$

where we used the integral representation (9) for the function ψ_2. □

Example 2.12. The most obvious example of exponentially convex function is $x \mapsto ce^{\alpha x}$, where $c \geq 0$ and $\alpha \in \mathbb{R}$ are constants.

Little less obvious examples can be deduced using integral representation (9) and some results from Laplace transformation.

Example 2.13. For every $\alpha > 0$, the function $\psi : (0, \infty) \to \mathbb{R}$ defined by

$$\psi(x) = x^{-\alpha}$$

is exponentially convex on $(0, \infty)$, since $x^{-\alpha} = \int_0^\infty e^{-xt} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt$ (see [15], p. 210).

Example 2.14. For every $\alpha > 0$, the function

$$\psi(x) = e^{-\alpha\sqrt{x}}$$

is exponentially convex on $(0, \infty)$, since $e^{-\alpha\sqrt{x}} = \int_0^\infty e^{-xt} e^{-\alpha^2/4t} \frac{\alpha}{2\sqrt{\pi t^3}} dt$, $x > 0$ (see [15], p. 214).

Further analytical properties and more examples of exponentially convex functions are contained in the following theorem from [4].

Theorem 2.15. *If $\psi : \mathbb{R} \to \mathbb{R}$ is an exponentially convex function, then it is entire, and $\varphi(t) = \psi(it)$, $t \in \mathbb{R}$ is a positive definite function. Conversely, if $\varphi : \mathbb{R} \to \mathbb{C}$ is an entire positive definite function, then $\psi(t) = \varphi(-it)$ is an exponentially convex function.*

Proof. See [4]. $\square$

Remark 2.16. (i) Recall that $\psi : \mathbb{R} \to \mathbb{C}$ is entire if it can be extended to a necessarily unique analytic function $\psi : \mathbb{C} \to \mathbb{C}$.

(ii) Conclusions of Theorem 2.15 can be extended to exponentially convex functions defined on any open interval (see [3] and [4]).

Using the previous theorem, the following examples (and many more) were constructed in [4].

Example 2.17. A characteristic function of the uniform distribution on $[0, 1]$ is $\varphi(t) = \frac{e^{it}-1}{it}$. Applying Theorem 2.15, we get exponentially convex function

$$\psi(x) = \frac{e^x - 1}{x}.$$

Example 2.18. A characteristic function of the normal distribution $N(\mu, \sigma^2)$; $\mu \in \mathbb{R}$, $\sigma > 0$; is $\varphi(t) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$. Applying Theorem 2.15, we get the exponentially convex function

$$\psi(x) = e^{-\mu x + \sigma^2 x^2}.$$

We now proceed with other properties of exponential convexity.

Theorem 2.19. *Assume that $\psi : I \to \mathbb{R}$ is an exponentially convex function on I. Then*

(*i*) *for any $k \in \mathbb{N}$ we have*

$$\psi^{(k)}(x) = \int_{-\infty}^{\infty} t^k e^{tx} d\sigma(t),$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is some non-decreasing function;

(*ii*) *for any $k \in \mathbb{N}$ the function $x \mapsto \psi^{(2k)}(x)$ is exponentially convex function on I;*

(*iii*) *for any $x \in I$ and all $k \in \mathbb{N}$ we have*

$$\left(\psi^{(2k+2)}(x)\right)^2 \leq \psi^{(2k)}(x)\psi^{(2k+4)}(x); \tag{10}$$

(*iv*) *for any $n \in \mathbb{N}$ and for all $x_i \in I$, $1 \leq i \leq n$, the matrix*

$$\left[\psi^{(i+j)}\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=0}^{n} \tag{11}$$

is positive-semidefinite.

Proof. (*i*) and (*ii*)-part follow by using integral representation (9). Proofs of (*iii*) and (*iv*) can be found in [5] and [10] respectively. □

Remark 2.20. From Theorem 2.19 it follows that if ψ is a non-constant exponentially convex function on I then $\psi^{(2k)}$ is also a non-constant exponentially convex function on I, for any $k \in \mathbb{N}$. Now, it is obvious that no polynomial can be exponentially convex.

Using the integral representation of an exponentially convex function, we will now give an extension of a result given by C. H. Kimberling in [7] and D. S. Mitrinović and J. E. Pečarić [10] for completely monotone functions.

Theorem 2.21. *Let $\psi : I \to \mathbb{R}$ be an exponentially convex function.*

(*i*) *If $0 \in I$, $x_i \in I$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} x_i \in I$, then for any natural number $m \geq 2$*

$$\psi^{m-1}(0)\psi\left(\sum_{i=1}^{m} x_i\right) \geq \prod_{i=1}^{m} \psi(x_i),$$

where for all $i = 1, \ldots n$, $x_i \geq 0$ or $x_i \leq 0$

(*ii*) *If $0 \in I$, and $x, y \in I$ such that $x \cdot y < 0$, $x + y \in I$ then*

$$\psi(0)\psi(x + y) \leq \psi(x)\psi(y)$$

(*iii*) *If $x \in I$, and i, j are odd natural numbers then*

$$\psi(x)\psi^{(i+j)}(x) \geq \psi^{(i)}(x)\psi^{(j)}(x).$$

Proof. *(i)*

$$\psi^{m-1}(0)\psi\left(\sum_{i=1}^{m} x_i\right) = \left(\int_{-\infty}^{\infty} \sigma(dt)\right)^{m-1} \int_{-\infty}^{\infty} \prod_{i=1}^{m} e^{tx_i}\sigma(dt)$$

We now apply Chebyshev's inequality (see [13], p. 197) for $p(t) = 1,\ f_i(t) = e^{tx_i},\ i = 1, \ldots, m$.

(ii) We apply Chebyshev's inequality for $p(t) = 1,\ f(t) = e^{tx},\ f(t) = e^{ty}$.

(iii)

$$\psi(x)\psi^{(i+j)}(x) = \int_{-\infty}^{\infty} e^{tx}\sigma(dt) \int_{-\infty}^{\infty} t^i t^j e^{tx}\sigma(dt).$$

We now apply Chebyshev's inequality for $p(t) = e^{tx}, f(t) = t^i, g(t) = t^j$. □

Remark 2.22. As we have seen in Corollary 2.5, every positive exponentially convex function on I is also log-convex. However, converse is not true: the function $h(x) = e^{x^3-x}$ is log-convex on $(0, 1)$ but it is not exponentially convex on $(0, 1)$. If h would be exponentially convex on $(0, 1)$ then, by Theorem 2.19, $h^{(4)}$ would also be exponentially convex on $(0, 1)$. But this is not true since, for example, $h^{(4)}(0.2) < 0$.

3. Main results-Generating method for exponential convexity

For $n \in \mathbb{N}$, let us denote by $K_n[a, b]$ all functions from $C[a, b]$ that are n−convex. Hence, $f \in K_n[a, b]$ if $[x_0, x_1, \ldots, x_n; f] \geq 0$ for any choice of mutually different numbers $x_i \in [a, b]$, $i = 0, \ldots, n$. Here $[x_0, x_1, \ldots, x_n; f]$ denotes divided difference of the function f in the knots $x_0, x_1, \ldots, x_n \in [a, b]$.

In the sequel, we consider linear functionals $A : C[a, b] \to \mathbb{R}$ that have the property

$$f \in K_n[a, b] \Rightarrow Af \geq 0. \tag{12}$$

Remark 3.1. Observe that $f \mapsto [x_0, x_1, \ldots, x_n; f]$ is one example of a linear functional that has property (12).

Theorem 3.2. *Let $A : C[a, b] \to \mathbb{R}$ be a linear functional such that it satisfies* (12) *and let I be any open interval in $\mathbb{R}$. Assume that $\mathbf{F} = \{f_t : t \in I\}$ is the family of functions from $C[a, b]$ such that $t \mapsto [x_0, x_1, \ldots, x_n; f_t]$ is a log-convex function in J-sense on I, for every choice of $n + 1$ mutually different numbers $x_0, x_1, \ldots, x_n \in [a, b]$. Then*

(i) the function $t \mapsto A(f_t)$ is also log-convex in J-sense on I;

(ii) if $t \mapsto A(f_t)$ is a continuous positive function on I, then the function $t \mapsto A(f_t)$ is log-convex on the I;

(*iii*) *if $t \mapsto A(f_t)$ is a positive, derivable function on I, then for any $p \leq u$, $q \leq v$, $p, q, u, v \in I$, we have*

$$M_{p,q}(A, \mathbf{F}) \leq M_{u,v}(A, \mathbf{F}) \tag{13}$$

where

$$M_{p,q}(A, \mathbf{F}) = \begin{cases} \left(\frac{A(f_p)}{A(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(A(f_p))}{A(f_p)}\right), & p = q. \end{cases} \tag{14}$$

Proof. (*i*) Choose any $r, s \in \mathbb{R}$ and $p, q \in I$. We define the function

$$h(x) = r^2 f_p(x) + 2rs f_{\frac{p+q}{2}}(x) + s^2 f_q(x).$$

Since log-convexity in J-sense of the function $t \mapsto [x_0, x_1, \ldots, x_n; f_t]$ is equivalent to positive definiteness of the quadratic form

$$\begin{aligned} &[x_0, x_1, \ldots, x_n; h] \\ &= r^2[x_0, x_1, \ldots, x_n; f_p] + 2rs[x_0, x_1, \ldots, x_n; f_{\frac{p+q}{2}}] + s^2[x_0, x_1, \ldots, x_n; f_q] \geq 0, \end{aligned}$$

we conclude $h \in K_n[a, b]$ and then $A(h) \geq 0$. Hence

$$r^2 A(f_p) + 2rsA(f_{\frac{p+q}{2}}) + s^2 A(f_q) \geq 0,$$

concluding log-convexity in J-sense of the function $t \mapsto A(f_t)$.

(*ii*) If $t \mapsto A(f_t)$ is additionally continuous, then it is log-convex since J-convex continuous functions are convex functions.

(*iii*) The conclusion of this part follows from Theorem 2.9. □

If we take the limit $x_i \to x$, $i = 0, 1, \ldots, n$, $x \in [a, b]$, from Theorem 3.2 we get the following corollary.

Corollary 3.3. *Let $A : C[a, b] \to \mathbb{R}$ be a linear functional such that it satisfies* (12) *and let I be any open interval in $\mathbb{R}$. Assume that $\mathbf{F} = \{f_t : t \in I\}$ is the family of $n-$times differentiable functions on $[a, b]$, such that $t \mapsto f_t^{(n)}(x)$ is a log-convex function in J-sense on I, for any $x \in \mathbb{R}$. Then*

(*i*) *the function $t \mapsto A(f_t)$ is also log-convex in J-sense on I;*

(*ii*) *if $t \mapsto A(f_t)$ is a continuous positive function on I, then the function $t \mapsto A(f_t)$ is log-convex on the I;*

(*iii*) *if $t \mapsto A(f_t)$ is a positive, derivable function on I, then for any $p \leq u$, $q \leq v$, $p, q, u, v \in I$, we have*

$$M_{p,q}(A, \mathbf{F}) \leq M_{u,v}(A, \mathbf{F}) \tag{15}$$

where

$$M_{p,q}(A, \mathbf{F}) = \begin{cases} \left(\frac{A(f_p)}{A(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{\frac{d}{dp}(A(f_p))}{A(f_p)}\right), & p = q. \end{cases} \tag{16}$$

Definition 3.4. $M_{p,q}(A,\mathbf{F})$ defined by (16) is called a mean if

$$a \leq M_{p,q}(A,\mathbf{F}) \leq b,$$

for $p,q \in I$.

Theorem 3.5. *Let $A : C[a,b] \to \mathbb{R}$ be a linear functional which satisfies* (12) *and let I be any open interval in $\mathbb{R}$. Assume that $\mathbf{F} = \{f_t : t \in I\}$ is the family of functions from $C[a,b]$ such that $t \mapsto [x_0, x_1, \ldots, x_n; f_t]$ is an exponentially convex function on I, for every choice of $n+1$ mutually different numbers $x_0, x_1, \ldots, x_n \in [a,b]$. Then*

(*i*) *if $t \mapsto A(f_t)$ is continuous on I, then it is also the exponentially convex function on I;*

(*ii*) *if $m \in \mathbb{N}$, $r_1, \ldots r_m \in I$ are arbitrary then $\left[A(f_{\frac{r_i+r_j}{2}})\right]_{i,j=1}^m$ is a positive semi-definite matrix. Particularly,*

$$\det\left[A\left(f_{\frac{r_i+r_j}{2}}\right)\right]_{i,j=1}^m \geq 0;$$

(*iii*) *if $t \mapsto A(f_t)$ is positive and continuous on I, then for any $p \leq u$, $q \leq v, p, q, u, v \in I$, we have*

$$M_{p,q}(A,\mathbf{F}) \leq M_{u,v}(A,\mathbf{F}) \tag{17}$$

where $M_{p,q}(A,\mathbf{F})$ is defined by (16).

Proof. (*i*) Choose any $m \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $p_i \in I$, for $i = 1, \ldots, n$.
We define

$$h(x) = \sum_{i,j=1}^{m} \xi_i \xi_j f_{\frac{p_i+p_j}{2}}(x).$$

By assumption, $[x_0, x_1, \ldots, x_n; h] = \sum_{i,j=1}^m \xi_i\xi_j [x_0, x_1, \ldots, x_n; f_{\frac{p_i+p_j}{2}}] \geq 0$ so $A(h) \geq 0$. Now, $\sum_{i,j=1}^m \xi_i\xi_j A(f_{\frac{p_i+p_j}{2}}) \geq 0$.

(*ii*) This is a consequence of Corollary 2.4.

(*iii*) Since $t \mapsto A(f_t)$ is derivable, we have our conclusion using Corollary 3.3(*iii*). □

Taking the limit $x_i \to x$, $i = 0, 1, \ldots, n$, $x \in [a,b]$, from Theorem 3.5 we get the following corollary

Corollary 3.6. *Let $A : C[a,b] \to \mathbb{R}$ be a linear functional that satisfies* (12) *and let I be any open interval in $\mathbb{R}$. Assume that $\mathbf{F} = \{f_t : t \in I\}$ is the family of $n-$time differentiable functions on $[a,b]$, such that $t \mapsto f_t^{(n)}(x)$ is an exponentially convex function on I, for any $x \in \mathbb{R}$. Then*

(*i*) *if $t \mapsto A(f_t)$ is continuous on I, then it is also exponentially convex function on I;*

(*ii*) *if* $m \in \mathbb{N}$, $r_1, \ldots r_m \in I$ *are arbitrary, then* $\left[A(f_{\frac{r_i+r_j}{2}})\right]_{i,j=1}^{m}$ *is a positive semi-definite matrix. Particularly,*

$$\det\left[A\left(f_{\frac{r_i+r_j}{2}}\right)\right]_{i,j=1}^{m} \geq 0;$$

(*iii*) *if* $t \mapsto A(f_t)$ *is positive and continuous on* I*, then for any* $p \leq u$, $q \leq v, p, q, u, v \in I$*, we have*

$$M_{p,q}(A,\mathbf{F}) \leq M_{u,v}(A,\mathbf{F}) \tag{18}$$

where $M_{p,q}(A,\mathbf{F})$ *is defined by* (16).

4. Application to means

In this section we will vary on the choice of a family $\mathbf{F} = \{f_t : t \in I\}$ in order to construct different examples of exponentially convex functions and means. At the beginning, we will need some general results.

Theorem 4.1. *Let* $f \in C^n[a,b]$ *and let* $A : C[a,b] \to \mathbb{R}$ *be a linear functional which satisfies property* (12). *Then there exists* $\xi \in [a,b]$ *such that*

$$A(f) = f^{(n)}(\xi)A(g_0), \tag{19}$$

where $g_0(x) = x^n/n!$.

Proof. Let $m = \min_{x\in[a,b]} f^{(n)}(x)$, $M = \max_{x\in[a,b]} f^{(n)}(x)$. Let us observe that function $\varphi(x) = M\frac{x^n}{n!} - f(x) = Mg_0(x) - f(x)$ is $n-$convex function since $\varphi^{(n)}(x) = M - f^{(n)}(x) \geq 0$. Hence, $A(\varphi) \geq 0$ and we conclude

$$A(f) \leq MA(g_0).$$

Similarly,

$$mA(g_0) \leq A(f) \leq MA(g_0).$$

Now we have (19). □

Corollary 4.2. *Let* $f, g \in C^n[a,b]$, *let* $A : C[a,b] \to \mathbb{R}$ *be a linear functional which satisfies* (12). *Then there exists* $\xi \in [a,b]$ *such that*

$$\frac{f^{(n)}(\xi)}{g^{(n)}(\xi)} = \frac{A(f)}{A(g)}, \tag{20}$$

assuming neither of the denominators is equal to zero.

We will now apply Corollary 4.2 in order to get a criteria for the recognition of means in the later part of the paper.

Corollary 4.3. *Let I be an open interval in $\mathbb{R}$, $a, b \in \mathbb{R}$ and $A : C[a,b] \to \mathbb{R}$ a linear functional which satisfies* (12). *Let $\mathbf{F} = \{f_t : t \in I\}$ be a family of functions in $C^n[a,b]$. If*

$$a \leq \left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}}(x) \leq b,$$

for $x \in [a,b]$, $p, q \in I$, then $M_{p,q}(A, \mathbf{F})$ is a mean.

Remark 4.4. In some examples that will follow, we will have very simple recognition of means:

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}}(x) = x, \quad x \in [a,b], \ p \neq q.$$

In the sequel, for our applications, we will take a n-th divided difference as a choice for the linear operator $A : C[a,b] \to \mathbb{R}$ that satisfies (12).

Example 4.5. Let a, b be positive real numbers, $I = (0, \infty)$ and family $\mathbf{F} = \{f_t : t \in I\}$ of functions defined on $C[a,b]$ with

$$f_t(x) = \begin{cases} \frac{t^{-x}}{(-\ln t)^n}, & t \neq 1; \\ \frac{x^n}{n!}, & t = 1. \end{cases} \tag{21}$$

Since $t \mapsto \frac{d^n}{dx^n} f_t(x) = t^{-x}$, by Example 2.13. we know that $t \mapsto \frac{d^n}{dx^n} f_t(x)$ is an exponentially convex function on $I = (0, \infty)$. From Corollary 3.6 we conclude exponential convexity of the function $\psi(t) = [x_0, x_1, \ldots, x_n; f_t]$ on $(0, \infty)$ for any choice of $n+1$ mutually different numbers $x_0, x_1, \ldots, x_n \in [a,b]$.

$M_{p,q}(\mathbf{x}, \mathbf{F})$ introduced in (16), in this particular case, take the form:

$$M_{p,q}(\mathbf{x}, \mathbf{F}) = \begin{cases} \left(\frac{[x_0, x_1, \ldots, x_n; f_p]}{[x_0, x_1, \ldots, x_n; f_q]}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{[x_0, x_1, \ldots, x_n; g_p]}{[x_0, x_1, \ldots, x_n; f_p]}\right), & p = q \neq 1; \\ \exp\left(-\frac{x_0 + x_1 + \cdots + x_n}{n+1}\right), & p = q = 1. \end{cases} \tag{22}$$

where $g_p(x) = \frac{d}{dp} f_p(x) = p^{-x-1}(-\ln p)^{-n-1}(n + \ln p)$.

Using monotonicity (18), from Corollary 3.6 we conclude

$$M_{p,q}(\mathbf{x}, \mathbf{F}) \leq M_{u,v}(\mathbf{x}, \mathbf{F}) \tag{23}$$

for any $p, q, u, v \in (0, \infty)$ such that $p \leq u$, $q \leq v$.
It is interesting to observe that after substitutions $x_i \to -\ln x_i$ in means $M_{1,1}(\mathbf{x}, \mathbf{F})$ we get the geometric mean of numbers $x_0, x_1, \ldots, x_n$:

$$M_{1,1}(-\ln \mathbf{x}, \mathbf{F}) = \sqrt[n+1]{x_0 x_1 \cdots x_n},$$

where $-\ln \mathbf{x} = (-\ln x_0, \ldots, -\ln x_n)$.

Example 4.6. Let a, b be positive real numbers, $I = (0, \infty)$ and family $\mathbf{F} = \{f_t : t \in I\}$ of functions defined on $C[a, b]$ with

$$f_t(x) = \frac{e^{-x\sqrt{t}}}{(-\sqrt{t})^n} \tag{24}$$

Since $\frac{d^n}{dx^n} f_t(x) = e^{-x\sqrt{t}}$, by Example 2.14, we know that $t \mapsto \frac{d^n}{dx^n} f_t(x)$ is an exponentially convex function on $I = (0, \infty)$. From Corollary 3.6 we conclude exponential convexity of the function $\psi(t) = [x_0, x_1, \dots, x_n; f_t]$ on $(0, \infty)$.
$M_{p,q}(\mathbf{x}, \mathbf{F})$ in this case are

$$M_{p,q}(\mathbf{x}, \mathbf{F}) = \begin{cases} \left(\frac{[x_0,x_1,\dots,x_n;f_p]}{[x_0,x_1,\dots,x_n;f_q]}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{[x_0,x_1,\dots,x_n;g_p]}{[x_0,x_1,\dots,x_n;f_p]}\right), & p = q. \end{cases} \tag{25}$$

where $g_p(x) = \frac{d}{dp} f_p(x) = \frac{(-1)^n}{2} p^{-\frac{n}{2}-1} e^{-x\sqrt{p}}(n + x\sqrt{p})$.
As in (23), we also have the monotonicity property.

Example 4.7. Let a, b real numbers, $I = \mathbb{R}$ and a family $\mathbf{F} = \{f_t : t \in I\}$ of functions defined by

$$f_t(x) = \begin{cases} \frac{e^{tx}}{t^n}, & t \neq 0; \\ \frac{x^n}{n!}, & t = 0 \end{cases} \tag{26}$$

Since $\frac{d^n}{dx^n} f_t(x) = e^{tx}$, $t \mapsto \frac{d^n}{dx^n} f_t(x)$ is an exponentially convex function by Example 2.12. From Corollary 3.6 we then conclude exponential convexity of the function $\psi(t) = [x_0, x_1, \dots, x_n; f_t]$.
We now deduce $M_{p,q}(\mathbf{x}, \mathbf{F})$.

$$M_{p,q}(\mathbf{x}, \mathbf{F}) = \begin{cases} \left(\frac{[x_0,x_1,\dots,x_n;f_p]}{[x_0,x_1,\dots,x_n;f_q]}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{[x_0,x_1,\dots,x_n;g_p]}{[x_0,x_1,\dots,x_n;f_p]}\right), & p = q; \\ \exp\left(\frac{x_0+x_1+\dots+x_n}{n+1}\right), & p = q = 0, \end{cases}$$

where $g_p(x) = p^{-n-1} e^{px}(xp^2 - n)$.
Monotonicity: if $p, q, u, v \in \mathbb{R}$ such that $p \leq u$, $q \leq v$, then

$$M_{p,q}(\mathbf{x}, \mathbf{F}) \leq M_{u,v}(\mathbf{x}, \mathbf{F}).$$

We observe here that $\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}}\right)^{\frac{1}{p-q}} (\ln x) = x$, so after the substitution $x_i \to \ln x_i$, $i = 0, 1, \dots, n$ in the above expressions we will get means for the numbers $x_0, x_1, \dots, x_n$:

$$M_{p,q}(\ln \mathbf{x}, \mathbf{F}) = \begin{cases} \left(\frac{[\ln x_0,\ln x_1,\dots,\ln x_n;f_p]}{[\ln x_0,x_1,\dots,\ln x_n;f_q]}\right)^{\frac{1}{p-q}}, & p \neq q; \\ \exp\left(\frac{[\ln x_0,\ln x_1,\dots,\ln x_n;g_p]}{[\ln x_0,\ln x_1,\dots,\ln x_n;f_p]}\right), & p = q; \\ \sqrt[n+1]{x_0 x_1 \cdots x_n}, & p = q = 0, \end{cases}$$

Example 4.8. Let a, b be positive real numbers, $I = \mathbb{R}$ and family $\mathbf{F} = \{f_t : t \in I\}$ of functions defined by

$$f_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0, 1, \ldots, n-1\}; \\ \frac{x^j \ln x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0, 1, \ldots, n-1\}. \end{cases} \tag{27}$$

Since $\frac{d^n}{dx^n} f_t(x) = x^{t-n} = e^{(t-n)\ln x}$; $t \mapsto \frac{d^n}{dx^n} f_t(x)$ is exponentially convex function by Example 2.12. From Corollary 3.6 we then conclude exponential convexity of the function $\psi(t) = [x_0, x_1, \ldots, x_n; f_t]$.

For this choice of family $\mathbf{F}$ we have

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}} \right)^{\frac{1}{p-q}} (x) = x, \quad x \in [a, b]$$

so using Remark 4.4 we have an important conclusion that $M_{p,q}(\mathbf{x}, \mathbf{F})$ is in fact a mean of numbers $x_0, x_1, \ldots, x_n \in [a, b]$.

In order to deduce all limit cases for these means we have to introduce some notation.

By $V(\mathbf{x}; f)$ we denote

$$V(\mathbf{x}; f) := \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & f(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & f(x_n) \end{vmatrix}$$

Particulary, for $f(t) = t^r \ln^k t$ we will denote

$$V(\mathbf{x}; r, k) := \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^r \ln^k x_0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^r \ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^r \ln^k x_n \end{vmatrix}.$$

Similarly, we denote

$$W(\mathbf{x}; r, k) := \begin{vmatrix} 1 & \ln x_0 & \ln^2 x_0 & \cdots & \ln^{n-1} x_0 & x_0^r \ln^k x_0 \\ 1 & \ln x_1 & \ln^2 x_1 & \cdots & \ln^{n-1} x_1 & x_1^r \ln^k x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \ln x_n & \ln^2 x_n & \cdots & \ln^{n-1} x_n & x_n^r \ln^k x_n \end{vmatrix},$$

Now,

$$[x_0, x_1, \ldots, x_n; f] = \frac{V(\mathbf{x}; f)}{V(\mathbf{x}; n, 0)}. \tag{28}$$

Means, completed with all expressions in limit cases:

$$M_{p,q}(\mathbf{x},\mathbf{F})=\begin{cases}\left(\frac{\prod_{k=0}^{n-1}(q-k)}{\prod_{k=0}^{n-1}(p-k)}\frac{V(\mathbf{x};p,0)}{V(\mathbf{x};q,0)}\right)^{\frac{1}{p-q}}, & (p-q)\prod_{k=0}^{n-1}[(q-k)(p-q)]\neq 0;\\ \left(\frac{\prod_{k=0}^{n-1}(q-k)}{(-1)^{n-1-j}j!(n-1-j)!}\,\frac{V(\mathbf{x};j,1)}{V(\mathbf{x};q,0)}\right)^{\frac{1}{j-q}}, & q\neq p=j\in\{0,1,\dots,n-1\};\\ \left((-1)^{k-j}\binom{n-1}{j}\binom{n-1}{k}^{-1}\frac{V(\mathbf{x};j,1)}{V(\mathbf{x};k,1)}\right)^{\frac{1}{j-k}}, & p=k\neq j=q,\ k,j\in\{0,1,\dots,n-1\};\\ \exp\left(\frac{V(\mathbf{x};q,1)}{V(\mathbf{x};q,0)}-\sum_{k=0}^{n-1}\frac{1}{q-k}\right), & p=q\notin\{0,1,\dots,n-1\};\\ \exp\left(\frac{V(\mathbf{x};q,2)}{2V(\mathbf{x};q,1)}-\sum_{\substack{k=0\\k\neq q}}^{n-1}\frac{1}{q-k}\right), & p=q\in\{0,1,\dots,n-1\}.\end{cases} \tag{29}$$

Means (29) are known in the literature as *generalized Stolarsky means* (see [9]). Using Corollary 3.6, we conclude that we have monotonicity of these means:

If $p,q,u,v\in\mathbb{R}$ such that $p\leq u,\ q\leq v$, then

$$M_{p,q}(\mathbf{x},\mathbf{F})\leq M_{u,v}(\mathbf{x},\mathbf{F}).$$

We observe here the following interesting property of the family $\mathbf{F}$ defined by (27): for any $x\in[a,b]$, the function $t\mapsto f_t(x)$ is not continuous in any of points $t\in\{0,1,\dots,n-1\}$. However, the function $t\mapsto A(f_t)$ will be continuous on $\mathbb{R}$ for any continuous linear functional $A:C[a,b]\to\mathbb{R}$ with property (12), as the next proposition shows.

Corollary 4.9. *If $A:C[a,b]\to\mathbb{R}$ is a continuous linear functional, then $t\mapsto A(f_t)$ is a continuous function on $\mathbb{R}$.*

Proof. The function $t\mapsto A(f_t)$ is obviously continuous in any point of the set $\mathbb{R}\backslash\{0,1,\dots,n-1\}$. Further, according to Theorem 4.1, $A(e_i)=0,\ i=0,1\dots,n-1$, where $e_i(t)=t^i$.
Let $j\in\{0,1,\dots,n-1\}$ be arbitrary. Then

$$x^t=x^j\left(1+(t-j)\ln x+o_{t-j}(x)\right)$$

where $\lim_{t\to j}\frac{\|o_{t-j}\|}{t-j}=0$. Now

$$A(f_t-f_j)=A(g)\left(\prod_{\substack{k=0\\k\neq j}}^{n-1}\frac{1}{t-k}-\frac{1}{(-1)^{n-1-j}j!(n-1-j)!}\right)+A(e_j\cdot o_{t-j})\prod_{k=0}^{n-1}\frac{1}{t-k} \tag{30}$$

where $g(x)=x^j\ln x$. Since $\|A(e_j\cdot o_{t-j})\|\leq M\|e_j\|\|o_{t-j}\|$ then both summands on the right hand side in (30) go to zero when $t\to j$. □

5. Generalized Pečarić-Šimić means

We now make one step further adding one more parameter in the generalized Stolarsky means. Let us take substitutions $p \to \frac{p}{r}$, $q \to \frac{q}{r}$, $x_i \to x_i^r$, $0 \le i \le n$ in the exponentially convex function $\psi(t) = [x_0, x_1, \ldots, x_n; f_t]$ from the generalized Stolarsky means.
We first show that after these substitutions the function:

$$\psi_r(t) = [x_0^r, x_1^r, \ldots, x_n^r; f_{\frac{t}{r}}] \tag{31}$$

is exponentially convex for any $r \neq 0$. Indeed, the function $t \mapsto [x_0^r, x_1^r, \ldots, x_n^r; f_t]$ is exponentially convex, so $t \mapsto [x_0^r, x_1^r, \ldots, x_n^r; f_{\frac{t}{r}}]$ is also exponentially convex according to Remark 2.3 (iii).
Now we proceed to the means. Since

$$\left(\frac{\frac{d^n f_{\frac{p}{r}}}{dx^n}}{\frac{d^n f_{\frac{q}{r}}}{dx^n}} \right)^{\frac{1}{p-q}} (x^r) = x, \quad x \in [a, b],$$

we conclude that we will get means $M_{p,q;r}(\mathbf{x}, \mathbb{F})$ for the numbers $x_0, x_1, \ldots, x_n$:

$$M_{p,q;r}(\mathbf{x}, \mathbb{F}) = \begin{cases} \left(\frac{\prod_{k=0}^{n-1}(q-rk)}{\prod_{k=0}^{n-1}(p-rk)} \frac{V(\mathbf{x}^r; \frac{p}{r}, 0)}{V(\mathbf{x}^r; \frac{q}{r}, 0)} \right)^{\frac{1}{p-q}}, & r(p-q)\prod_{k=0}^{n-1}[(q-rk)(p-rk)] \neq 0; \\ \left(\frac{\prod_{k=0}^{n-1}(q-rk)}{r^n(-1)^{n-1-j} j!(n-1-j)!} \frac{V(\mathbf{x}^r; j, 1)}{V(\mathbf{x}^r; \frac{q}{r}, 0)} \right)^{\frac{1}{jr-q}}, & \begin{matrix} r(p-q)\prod_{k=0}^{n-1}[(q-rk)(p-rk)] \neq 0, \\ p = jr, \quad j \in \{0, 1, \ldots, n-1\} \end{matrix}; \\ \left((-1)^{k-j} \binom{n-1}{j} \binom{n-1}{k}^{-1} \frac{V(\mathbf{x}^r; j, 1)}{V(\mathbf{x}^r; k, 1)} \right)^{\frac{1}{(j-k)r}}, & \begin{matrix} r(p-q) \neq 0,\ p = jr,\ q = kr, \\ 0 \le j \neq k \le n-1 \end{matrix}; \\ \exp\left(\frac{V(\mathbf{x}^r; \frac{q}{r}, 1)}{rV(\mathbf{x}^r; \frac{q}{r}, 0)} - \sum_{k=0}^{n-1} \frac{1}{q-kr} \right), & p = q,\ r\prod_{k=0}^{n-1}(q-rk) \neq 0; \\ \exp\left(\frac{V(\mathbf{x}^r; j, 2)}{2V(\mathbf{x}^r; j, 1)} - \sum_{\substack{k=0 \\ k \neq j}}^{n-1} \frac{1}{q-kr} \right), & r \neq 0,\ p = q = jr,\ j \in \{0, 1, \ldots, n-1\}; \\ \left(\left(\frac{q}{p}\right)^n \frac{W(\mathbf{x}; p, 0)}{W(\mathbf{x}; q, 0)} \right)^{\frac{1}{p-q}}, & pq(p-q) \neq 0,\ r = 0; \\ \left(\frac{n!}{q^n} \frac{W(\mathbf{x}; q, 0)}{W(\mathbf{x}; 0, n)} \right)^{\frac{1}{q}}, & q \neq 0,\ p = r = 0; \\ \exp\left(\frac{W(\mathbf{x}; q, 1)}{W(\mathbf{x}; q, 0)} - \frac{n}{q} \right), & p = q \neq 0,\ r = 0; \\ \sqrt[n+1]{x_0 \cdot x_1 \cdots x_n}, & p = q = r = 0, \end{cases}$$

where $\mathbf{x}^r = (x_0^r, x_1^r, \ldots, x_n^r)$. These means were first considered in [20] as generalized *Pečarić-Šimić means.* Here we prove the monotonicity of the generalized Pečarić-Šimić means (property which was not given in [20]).

Theorem 5.1. *Let $p \le u$, $q \le v$. Then*

$$M_{p,q;r}(\mathbf{x}, \mathbb{F}) \le M_{u,v;r}(\mathbf{x}, \mathbb{F}), \tag{32}$$

for any $r \in \mathbb{R}$ and for all $x_0, x_1, \ldots, x_n$ mutually different, positive real numbers.

Proof. *Case* $r \neq 0$. Since the function $t \mapsto \psi_r(t) = [x_0^r, x_1^r, \ldots, x_n^r; f_{\frac{t}{r}}]$ is exponentially convex using (*iii*)-part of Theorem 3.6 we conclude monotonicity in this case.

Case $r = 0$. Since

$$M_{p,q;0}(\mathbf{x}, \mathbf{F}) = M_{p,q}(\ln \mathbf{x}, \mathbf{G})$$

where $\mathbf{G} = \{g_t : t \in I\}$ is family of exponentially convex functions from Example 4.7 our proof is done. □

6. Conclusion remarks

As we showed in previous examples, basic ingredients for the construction of exponentially convex functions is a family $\mathbf{F} = \{f_t : t \in I\}$ which have the property from Corollary 3.6 and a linear functional $A : C[a, b] \to \mathbb{R}$ which have the property (12).

Although divided difference is very natural choice for a linear functional A, it is pretty easy to find (construct) other examples for A from well-known inequalities:

if $n = 1$, the $K_1[a, b]$ is a family of increasing functions from $C[a, b]$ and using *Steffensen* inequality we can define a linear functional

$$A(f) = \int_{b-\lambda}^{b} f(t)dt - \int_{a}^{b} f(t)g(t)dt,$$

(g is integrable on $[a, b]$, $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$) to conclude that $t \mapsto A(f_t)$ is an exponentially convex function where

$$f_t(x) = \begin{cases} x^t/t, & t \neq 0; \\ \ln x, & t = 0 \end{cases}$$

if $n = 2$, the $K_2[a, b]$ is a family of convex functions from $C[a, b]$, and using *Jensen* inequality we can define a linear functional

$$B(f) = \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)$$

($x, y \in [a, b]$ are fixed) and a family $\mathbf{F}$

$$f_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 0, 1; \\ -\ln x, & t = 0; \\ x \ln x, & t = 1. \end{cases} \tag{33}$$

to conclude exponential convexity of the function $t \mapsto B(f_t)$.

Now it is clear that any inequality that is expressible in terms of linear functionals and which is valid on $K_n[a, b]$, for some $n \in N$, can be used for generating exponential convexity.

References

[1] N. I. Akhiezer: The Classical Moment Problem and Some Related Questions in Analysis, Oliver and Boyd, Edinburgh (1965).

[2] S. N. Bernstein: Sur les fonctions absolument monotones, Acta Math. 52 (1929) 1–66.

[3] T. M. Bisgaard, Z. Sasvári: Characteristic Functions and Moment Sequences, Positive Definiteness in Probability, Nova Science, Huntington (2000).

[4] W. Ehm, M. G. Genton, T. Gneiting: Stationary covariances associated with exponentially convex functions, Bernoulli 9(4) (2003) 607–615.

[5] C. Giordano, B. Palumbo, J. Pečarić: Remarks on the Hankel determinantal inequalities, Rend. Circ. Mat. Palermo, II. Ser. 46 (1997) 279–286.

[6] J. Jakšetić, J. Pečarić, Atiq ur Rehman: On Stolarsky and related means, Math. Inequal. Appl. 13 (2010) 899–909.

[7] C. H. Kimerling: A probabilistic interpretation of complete monotonicity, Aquationes Math. 10 (1974) 152–164.

[8] E. B. Leach, M. C. Sholander: Extended mean values, Amer. Math. Monthly 85 (1978) 84–90.

[9] E. B. Leach, M. C. Sholander: Multi-variable extended mean values, J. Math. Anal. Appl. 104 (1984) 390–407.

[10] D. S. Mitrinović, J. E. Pečarić: On some inequalities for monotone functions, Boll. Unione Mat. Ital., VII. Ser., B 2 (1991) 407–416.

[11] D. S. Mitrinović, J. Pečarić, A. M. Fink: Classical and New Inequalities in Analysis, Kluwer, Dordrecht (1993).

[12] C. P. Niculescu, L.-E. Persson: Convex Functions and their Applications. A Contemporary Approach, CMS Books in Mathematics 23, Springer, New York (2006).

[13] J. E. Pečarić, F. Proschan, Y. C. Tong: Convex Functions, Partial Orderings and Statistical Applications, Academic Press, New York (1992).

[14] T. Popoviciu: Notes sur les fonctions convexes dordre superieur (IX), Bull. Math. Soc. Roum. Sci. 43 (1941) 85–141.

[15] J. L. Schiff: The Laplace Transform. Theory and Applications, Undergraduate Texts in Mathematics, Springer, New York (1999).

[16] K. B. Stolarsky: Generalizations of the logarithmic mean, Math. Mag. 48 (1975) 87–92.

[17] K. B. Stolarsky: The power and generalized logarithmic means, Amer. Math. Monthly 87 (1980) 545–548.

[18] D. V. Widder: Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral, Trans. Amer. Math. Soc. 40 (1934) 321–326.

[19] D. V. Widder: The Laplace Transform, Princeton Univ. Press, New Jersey (1941).

[20] Zhen-gang Xiao, Zhi-hua Zhang, V. Lokesha, Rong Tang: Two-parameters the mean of n variables, Int. Rev. Pure Appl. Math. 1(1) (2005) 93–111.